



# Portfolio Allocation: An Empirical Analysis of Ten American Stocks for the Period 2010-2015

Emi Malaj<sup>1</sup>, Visar Malaj<sup>2</sup>

<sup>1</sup>University of Vlora, Faculty of Economy, Department of Economics, Vlora, Albania

<sup>2</sup>University of Tirana, Faculty of Economy, Department of Economics, Tirana, Albania

## Email address:

[emi.malaj@univlora.edu.al](mailto:emi.malaj@univlora.edu.al) (E. Malaj), [visarmalaj@feut.edu.al](mailto:visarmalaj@feut.edu.al) (V. Malaj)

## To cite this article:

Emi Malaj, Visar Malaj. Portfolio Allocation: An Empirical Analysis of Ten American Stocks for the Period 2010-2015. *International Journal of Accounting, Finance and Risk Management*. Vol. 1, No. 1, 2016, pp. 11-18. doi: 10.11648/j.ijafmr.20160101.12

**Received:** August 31, 2016; **Accepted:** October 20, 2016; **Published:** November 8, 2016

**Abstract:** We invest in order to obtain excess returns on the investment. The excess return is calculated with respect to the risk free rate and implies taking on risk. It is necessary to quantify both return and risk on the portfolio level. Asset returns are correlated, and for this reason the correlation matrix is estimated to quantify precisely the correlation among the returns on portfolio assets. These coefficients will then enable us to describe the combined returns on the portfolio's assets, and the risk of the portfolio. The basis of all quantitative portfolio management and theory today are given by the well-known Modern Portfolio Theory. We analyze in this paper ten American stocks from completely different industry sectors, part of the Standard & Poor's 500 index. The period is from December 2010 to December 2015, monthly observations. Since the end of 1999, the S&P's 500 stock index has lost an average of 3.3% a year on an inflation adjusted basis, compared with a 1.8% average annual gain during the 1930s when deflation afflicted the economy. In nearly 200 years of recorded stock-market history, no calendar decade has seen such a bad performance as the 2000s. The computer programs used in this work are MATLAB and Microsoft Excel. For optimization problems we mainly use the Excel Solver.

**Keywords:** Portfolio Allocation, Stock Prices, Standard and Poor's 500

## 1. Introduction

A portfolio in the financial econometrics field is a collection of some financial assets or instruments described by a list of the quantities or proportions of the total money amount invested in a limited number of these assets. The amount invested in a share (and the proportion) can be negative; this is called a short position. The main objectives of the portfolio theory are connected to the expected return and risk, estimated by the standard deviation. Portfolios are evaluated in terms of return and risk to find out which are "good" portfolios to hold, so where we can invest. This problem can be solved through two alternative ways: choosing the portfolio with the highest expected return for any given level of risk or with the lowest standard deviation for any given level of expected return. The set of the possible portfolios with three or more assets form a region. Portfolios which lie on the left hand boundary of this region are called minimum variance portfolios. The portfolio with the smallest variance of all is called the global minimum variance

portfolio. Minimum variance portfolios which lie above the global minimum variance portfolio are called efficient minimum variance portfolios.

The portfolio theory originally dates back to 1952, when Harry Markowitz published his article on what he called '*portfolio selection*'. In this article he established a framework for describing portfolios of assets in terms of the means on their returns, the variance of their returns, and the correlation between the returns on assets. For this reason the approach is also known as Mean-Variance Analysis. Since its formulation half a century ago it has been seized on by the investment industry as a workable tool for investment and risk management, in particular because of its simplicity and intuitive appeal, and it remains one of the cornerstones in the foundation on which today's asset management industry rests. There are, however, several potentially significant problems that one must be aware of when employing modern portfolio theory.

Tobin (1958) and Sharpe (1966) also contributed by extending and modifying the original theory of Markowitz. According to Affleck-Graves and Money (1976), the results

of the model of Sharpe (1966) enhance in a progressive manner with the addition of indexes. Mandelbrot (2004) concluded that, under some specific conditions, the Markowitz approach produces unrealistic portfolios, not well-diversified. Many other well-known authors have applied and demonstrated the Markowitz theory such as Bodie, Kane and Marcus (2005), Low, Faff, and Aas (2016) etc.

## 2. Diversification and Some Other Important Concepts

The risk of a portfolio, quantified by its volatility, is heavily dependent on the exact nature and magnitude of the covariance or correlation between asset returns. If the returns on assets in the portfolio are correlated, there may exist opportunities for reducing the level of total portfolio risk by selecting appropriate assets and asset weights, in an attempt to offset individual asset risks against each other. In other words we attempt, in a structured manner, to exploit the fact that asset returns often move in somewhat consistent patterns relative to each other. For example, if demand for autos is high and the stocks of auto manufacturers move up, so will the stocks of suppliers of auto parts, because they both depend on the end demand for autos. At least, that is what would happen in an ideal world. The returns on the two stocks are positively correlated, which means there is no significant benefit from diversification by holding both types of stock. However, holding auto stocks along with stocks of utilities companies, which are somewhat more defensive in nature, and thus tend to move down whenever consumer-oriented stocks such as autos move up, will most likely result in reduced variation in total portfolio return, simply because the returns on the two types of stocks tend to cancel each other out. This is the essence of diversification.

Portfolio risk is a quantity that requires the determination of the asset-by-asset correlations, as well as of individual asset volatilities. The portfolio volatility can be estimated using the standard deviation. The core of a portfolio's volatility is the variance-covariance matrix, which expresses all asset-by-asset covariances. The exact determination of portfolio volatility, given a set of assets and their individual characteristics, enables us to investigate the nature of portfolio diversification. With appropriate use of correlations between individual assets, it is in some cases possible to reduce portfolio volatility to levels below individual asset volatilities. It can be demonstrated that adding stocks to a portfolio will reduce its volatility, and as the number of assets goes towards infinity we obtain the market volatility.

In a portfolio context, efficiency is defined as the maximum attainable return for a given level of volatility, or alternatively, the minimum attainable volatility for a given level of return.

Thus, the mean-variance efficiency is used as the measure of portfolio efficiency, and more precisely we designate efficient portfolios as those portfolios that cannot be

improved upon in terms of the return versus risk trade-off. It is thus not possible to alter an efficient portfolio without paying a price in the form of lower return or higher volatility. The vast majority of attainable portfolios are not efficient in the strict mean-variance sense, which suggests that we can improve on them at no cost (in terms of return or volatility) by altering their composition. In other words, by changing the composition or weightings of the portfolios we should be able either to increase the level of expected return without lowering the expected level of risk, or to lower the expected level of risk without incurring a lower level of expected return. In general, quantitative portfolio optimization seeks to first determine:

- the minimum - variance portfolio
- the minimum level of volatility for a given level of return
- the maximum level of return for a given level of volatility.

Although on the surface these appear to be different goals, they essentially involve the same type of procedure. As noted, determining the asset weights that enable the creation of portfolios with the three characteristics listed above is achieved through quantitative portfolio optimization using mathematical optimization techniques.

The minimum-variance portfolio is the portfolio (that is the combination of asset weights) that, given the particular return and risk characteristics of each asset, generates the lowest amount of risk achievable. In other words, the minimum variance portfolio specifies the asset weights that generate the lowest possible portfolio risk, without any additional constraints on the desired level of return or on the maximum or minimum extent to which an asset can enter into the portfolio. The minimum variance portfolio is important in a portfolio analysis context because it reveals the absolute lowest level of portfolio risk that is attainable from the assets available to the portfolio manager. This level of risk may well end up being lower than the volatility of the market, because of the diversification effects discussed above, and also because here we have the option of dropping stocks from the portfolio whose characteristics do not meet our needs for diversification. So if the inclusion of an asset does not benefit the portfolio in terms of reduced overall portfolio volatility, we simply leave it out. The minimum-variance portfolio thus serves as a measure for assessing the properties of the portfolio in terms of diversification potential, and as a starting point for additional analysis of the range of mean-variance efficient portfolios that a particular set of assets is capable of generating.

The efficient frontier is the line between the minimum-variance portfolio and the maximum-variance portfolio that traces out all attainable portfolios (asset combinations) that produce optimal/efficient portfolios. In other words, the efficient frontier is the line in return/risk space that traces out all the portfolios for which we cannot obtain a higher level of return for a given level of risk, or alternatively for which we cannot obtain a lower level of risk for a given level of return.

We do not actually need to calculate all the possible

efficient portfolios to be able to produce the efficient frontier, which in principle contains an infinite number of different portfolios. In practice, computerized optimization techniques are employed, and so we can easily obtain as many points on the efficient frontier as we like.

### 3. Application

The empirical analysis in this section includes ten well known stocks, part of the U.S. market. The concepts described in the previous sections have been applied to different selected stock prices. First, we determine the returns from the selected stock prices (*Total Return Data*). Using the returns, we calculate the variance-covariance and the correlation matrices useful for the efficient frontier construction. Then, 5 stocks of the initial 10 have been chosen, in order to determine the risky asset Efficient Frontier without imposing any short selling constraint. Stocks have been selected so that to determine the “best” Efficient Frontier. The index Market Portfolio related to the stocks have been analyzed if it is efficient with respect to the Efficient Frontier, in order to define and construct the Global Minimum Portfolio based on our 5 stocks. Finally, we determine the expected return and the standard deviation of the equally weighted portfolio generated with the 5 stocks and the equally weighted portfolio generated with the 10 stocks. The expected returns of these two portfolios are compared with those of the portfolios in the Efficient Frontier having the same standard deviation or risk.

Let’s initially make some brief considerations about the U.S. stock market trend of the last years. The U.S. stock

market has been a tough place to make money during the last decade. With the S&P 500 stock index closing at 1180,70 (November 30, 2010), it is trading now below the 1362.80 of April 1999. The Dow Jones index is slightly better off, having risen about 1% a year since January 2000. For the past nine years, the S&P 500 is the worst-performing among nine different asset classes, including commodities, real-estate investment trusts, gold and foreign stocks. Even low-risk Treasury bonds have outperformed U.S. stocks. Once you adjust for inflation, there is little doubt that investors in the U.S. stock market have lost money in real terms. And with the relentless decline of the U.S. dollar during the past five years, foreign investors in U.S. stocks have fared substantially worse.

Stocks have been chosen from completely different industry sectors in order to differentiate as much as possible my portfolio. All the selected stocks are part of the S&P 500 index. The period is from December 31, 2010 to December 31, 2015 (monthly observations).

Let’s transform now prices in returns (in percentages) using:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1 \quad (1)$$

where  $P_t$  is the price at time  $t$  and  $R_t$  is the return at time  $t$ . Table 1 shows the return sample mean and standard deviation for each price series. The standard deviation, given as the square root of the variance, is a good estimator of the volatility of the process.

**Table 1.** Expected return and standard deviation.

Stocks	HASBRO	COCA COLA	JOHNSON & JOHNSON	HOST HOTELS & RESORTS	EXXON MOBIL
Exp. Return	0,0161	0,0157	0,0055	0,0113	0,0099
Standard Deviation	0,0715	0,0365	0,0765	0,1276	0,0465
Stocks	EDISON INTL	WHIRLPOOL	TITANIUM METALS	BANK OF AMERICA CORP.	RED HAT
Exp. Return	0,0088	0,0235	0,0431	0,0003	0,0365
Standard Deviation	0,0568	0,1638	0,1234	0,1886	0,1497

As we can see, all the expected returns are positive. The corresponding parameters for the S&P index are 0,5 basis points (expected return) and 43 basis points (standard deviation). Table 2 shows the variance-covariance matrix for the ten selected stocks. All the relations between the assets are positive except the relation between ‘Whirlpool’ and ‘Exxon Mobil’ asset that is negative. The covariance between these two stocks is -0,00092.

**Table 2.** Covariance matrix.

Stocks	HASBRO	COCA COLA	JOHNSON & JOHNSON	HOST HOTELS & RESORTS	EXXON MOBIL	EDISON INTL	WHIRLPOOL	TITANIUM METALS	BANK OF AMERICA CORP.	RED HAT
HASBRO	0,00598	0,00095	0,00252	0,00592	0,00229	0,00259	0,00599	0,0022	0,00529	0,00252
COCA COLA	0,00095	0,00269	0,00225	0,00299	0,00099	0,00095	0,00229	0,0029	0,00559	0,00099
JOHNSON & JOHNSON	0,00252	0,00225	0,00392	0,00205	0,00059	0,00099	0,00299	0,00095	0,00592	0,00205
HOST HOTELS & RESORTS	0,00592	0,00299	0,00205	0,02452	0,00059	0,00222	0,02592	0,02295	0,02525	0,00599
EXXON MOBIL	0,00229	0,00099	0,00059	0,00059	0,00579	0,00229	-0,00092	0,00292	0,00055	0,00052
EDISON INTL	0,00259	0,00095	0,00099	0,00222	0,00229	0,00528	0,00209	0,00252	0,005	0,00255
WHIRLPOOL	0,00599	0,00229	0,00299	0,02592	-0,00092	0,00209	0,02993	0,00595	0,02929	0,0059
TITANIUM METALS	0,0022	0,0029	0,00095	0,02295	0,00292	0,00252	0,00595	0,01951	0,00599	0,00999
BANK OF AMERICA CORP.	0,00529	0,00559	0,00592	0,02525	0,00055	0,005	0,02929	0,00599	0,05013	0,00525
RED HAT	0,00252	0,00099	0,00205	0,00599	0,00052	0,00255	0,0059	0,00999	0,00525	0,02211

Table 3 shows the correlation matrix of the ten selected assets. Almost all the stocks are positively correlated; the only negative correlation is between ‘Whirlpool’ and Exxon mobil’, equal to -0,08879.

Table 3. Correlation matrix.

Stocks	HASBRO	COCA COLA	JOHNSON & JOHNSON	HOST HOTELS & RESORTS	EXXON MOBIL	EDISON INTL	WHIRLPOOL	TITANIUM METALS	BANK OF AMERICA CORP.	RED HAT
HASBRO	1	0,33683	0,53363	0,50188	0,36118	0,36399	0,53838	0,16566	0,50168	0,3156
COCA COLA	0,33683	1	0,58335	0,31355	0,3533	0,30665	0,33361	0,30851	0,50303	0,10668
JOHNSON & JOHNSON	0,53363	0,58335	1	0,31631	0,1885	0,53661	0,51535	0,11083	0,55068	0,16653
HOST HOTELS & RESORTS	0,50188	0,31355	0,31631	1	0,05088	0,35355	0,66368	0,53855	0,5536	0,30168
EXXON MOBIL	0,36118	0,3533	0,1885	0,05088	1	0,35083	-0,08868	0,36818	0,05536	0,0586
EDISON INTL	0,36399	0,30665	0,53661	0,35355	0,35083	1	0,3618	0,36588	0,30685	0,3856
WHIRLPOOL	0,53838	0,33361	0,51535	0,66368	-0,08868	0,3618	1	0,30655	0,66368	0,33536
TITANIUM METALS	0,16566	0,30851	0,11083	0,53855	0,36818	0,36588	0,30655	1	0,30355	0,36851
BANK OF AMERICA CORP.	0,50168	0,50303	0,55068	0,5536	0,05536	0,30685	0,66368	0,30355	1	0,30366
RED HAT	0,3156	0,10668	0,16653	0,30168	0,0586	0,3856	0,33536	0,36851	0,30366	1

We choose now 5 of the 10 initial assets in order to determine the best efficient frontier. There are different approaches that can be used in order to select the best assets. One of these is to take the assets with the highest Sharp Ratio. Another way we can use is to take the assets that are less risky than the others, so with a lower variance, fixing a constant level of expected return and vice versa. The first approach has been followed, so we calculate the Sharpe Ratios for each stock.

So let's initially see an expected return –standard deviation plot of the ten risky assets. Figure 1 shows the level of risk associated to the expected return for each asset. We can say for example, that is much better investing in the ‘Titanium Metals’ asset than in the ‘Bank of America’ asset because the first asset has a bigger expected return and a lower risk represented by the standard deviation.

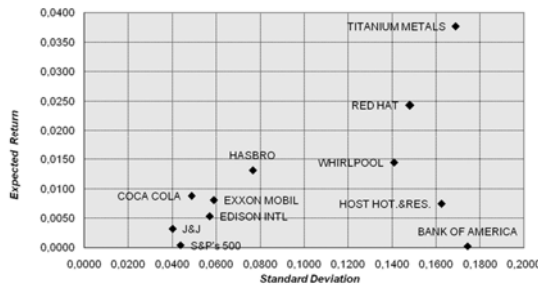


Figure 1. An expected return – standard deviation plot.

The asset performances can be evaluated through the the Sharp Ratio histogram. This is the Sharp ratio formula:

$$\text{SharpeRatio} = \frac{\bar{R} - r_f}{\sigma} \quad (2)$$

where  $\bar{R}$ ,  $r_f$  and  $\sigma$  are the expected return, the risk free rate and the standard deviation. We can say by the Sharp Ratio

histogram (figure 2) that the best five risky assets are: Hasbro, Coca Cola, Exxon, Titanium Metals and Red Hat. So these are my five selected asset.

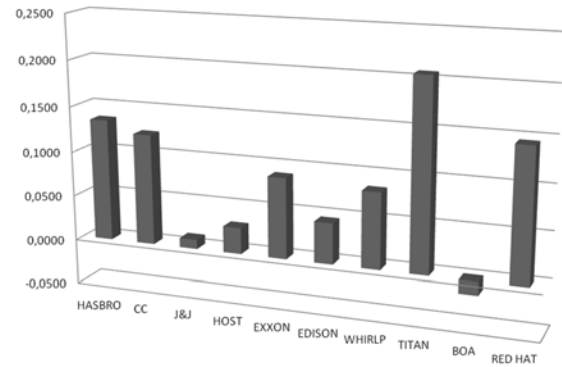


Figure 2. Sharpe ratio histogram.

Now we determine the efficient frontier for these five assets without imposing any short selling constraint. So the problem is to minimize risk for a given level of expected return.

$$\begin{aligned} \min_{\omega} \quad & \omega' \Sigma \omega \\ \text{s.t.} \quad & \omega' \mu = \mu_p \\ & \omega' \mathbf{1} = 1 \end{aligned} \quad (3)$$

where  $\omega$  is the portfolio weights vector,  $\Sigma$  is the variance-covariance matrix and  $\mu$  is the returns vector. The solution is:

$$\begin{aligned} \omega_k &= \frac{C\mu_p - B}{AC - B^2} \Sigma^{-1} \mu + \frac{A - B\mu_p}{AC - B^2} \Sigma^{-1} \mathbf{i} \\ A &= \mu' \Sigma^{-1} \mu \quad B = \mu' \Sigma^{-1} \mathbf{i} \quad C = \mathbf{i}' \Sigma^{-1} \mathbf{i} \\ \omega_k &= D + E\mu_p = \frac{A\Sigma^{-1} \mathbf{i} - B\Sigma^{-1} \mu}{AC - B^2} + \frac{C\Sigma^{-1} \mu - B\Sigma^{-1} \mathbf{i}}{AC - B^2} \mu_p \end{aligned} \quad (4)$$

where  $\mathbf{i}$  is the unitary vector. We can see the values of the

components in the table below.

**Table 4.** Computed parameters.

COMPONENTS	
A	0,0883
B	5,4845
C	592,956
D	22,233

The equation of the Efficient Frontier will be:

$$\sigma_p = \sqrt{\frac{C\mu_p^2 - 2B\mu_p + A}{AC - B^2}} \quad (5)$$

These are the first seven points that we use in order to draw the efficient frontier:

**Table 5.** Seven efficient frontier points.

Efficient Frontier Points	
Return	Standard Deviation
-0,05100	0,30345454
-0,04700	0,30345654
-0,04100	0,29454657
-0,04500	0,29456657
-0,04200	0,28566768
-0,04200	0,28565678
-0,04700	0,27566788

The Global Minimum Variance (GMV) portfolio is a fully-invested portfolio with the minimum volatility value. As mentioned before, the volatility can be estimated by the standard deviation. The GMV portfolio belongs to efficient frontier and is located on its left end. These will be the parameters for the GMV portfolio.

$$\omega_V = \frac{\Sigma^{-1}\mathbf{i}}{\mathbf{i}\Sigma^{-1}\mathbf{i}} \quad \sigma_V = \frac{1}{\sqrt{C}} \quad \mu_V = \frac{B}{C} \quad (6)$$

The Tangent Portfolio combines this optimal combination of risky assets with a risk-free asset. It has the highest Sharp Ratio. These will be the parameters for the Tangent Portfolio.

$$\omega_E = \frac{\Sigma^{-1}\mu}{\mathbf{i}\Sigma^{-1}\mu} \quad \sigma_V = \frac{\sqrt{A}}{B} \quad \mu_V = \frac{A}{B} \quad (7)$$

Table 6 shows the corresponding weights for each asset for both the Tangent and the Global Minimum Variance portfolio. We have calculated the portfolio standard deviation and return in each case.

**Table 6.** GMV and Tangent Portfolio parameters.

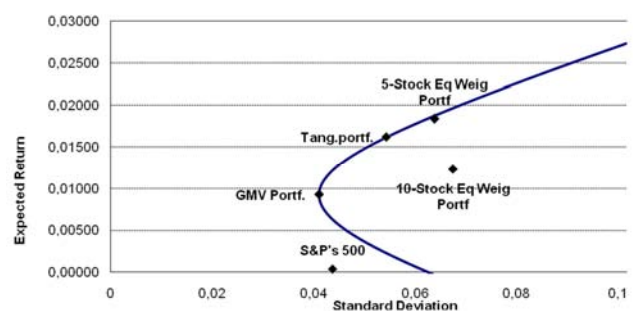
Portfolio	GMV	Tangent
HASBRO	0,143453	0,224544
COCA COLA	0,534457	0,395667
EXXON M.	0,305667	0,098788
TITANIUM M.	-0,026767	0,162113
RED HAT	0,043444	0,106778
Sum of Weights	1	1
Standard deviation	0,0414545	0,05454544
Return	0,0091345	0,01454566

We determine now the expected return and the standard deviation of the equally weighted portfolio generated with the 5 selected stocks and the equally weighted portfolio generated with the 10 stocks. An equally weighted portfolio would have the same amount of money invested in each unique stock. Therefore, the number of shares of each stock would be different, with more shares of cheaper stocks. An equally weighted portfolio would have to be rebalanced more frequently to maintain equal weight, because stocks prices would diverge quickly. Table 7 shows the portfolio standard deviation and return for the two equally weighted portfolios.

**Table 7.** Return and standard deviation for the two equally weighted portfolios.

Portfolio	5-Stock Equally Weighted	10-Stock Equally Weighted
Standard deviation	0,06334244	0,067342
Return	0,01843457	0,012343

Figure 3 shows the graphical representation of the Efficient Frontier and other relevant portfolios. The Efficient Frontier includes all the efficient portfolios. There are no portfolios with the same standard deviation and a greater return and vice versa. All the rational agents will choose their portfolio in this curve (tangency point with the indifference curves). The Market Index Portfolio is as expected on the left side of the efficient frontier. The Market Index Portfolio is composed by 500 risky assets including the our five assets of the efficient frontier. As we can see by the graph we can obtain a greater expected return than the S&P portfolio's one without changing the level of standard deviation. This can be possible by moving up vertically the Index Portfolio until we reach the Efficient Frontier. So it exists an Efficient Frontier portfolio more efficient that the Index Portfolio. We reach the same conclusion for the two equally weighted portfolios. So, the two equally weighted portfolios have a lower expected return than the efficient frontier portfolios with the same standard deviation.



**Figure 3.** Efficient frontier and some relevant portfolios.

Now we will see the same portfolio compositions imposing the short selling constraint. In finance, short selling (also known as shorting or going short) is the practice of selling assets, usually securities, that have been borrowed from a third part (usually a broker) with the intention of buying identical assets back at a later date to return to the lender. So, the problem in this case is:

$$\begin{aligned}
& \min_{\omega} \omega' \Sigma \omega \\
& s.t. \quad \omega' \mu = \mu_p \\
& \omega' \mathbf{1} = 1 \\
& \omega \geq 0
\end{aligned} \tag{8}$$

The excel solver has been used in order to draw the efficient frontier with the short selling constraint. We have used 23 points and we have in the table below 7 of them.

Table 8. Efficient frontier points.

HASBRO	0	0,05033	0,10234	0,14967	0,16676
COCA COLA	0,05334	0,58567	0,56569	0,52248	0,49454
EXXON MOBIL	0,94347	0,36568	0,32126	0,27567	0,24138
TITANIUM METALS	0	0	0	0	0,03467
RED HAT	0	0	0	0,04745	0,06245
Sum of Weights	1	1	1	1	1
Optimal Portfolio Return	0,0084	0,00894	0,009005	0,009995	0,011566
Target Portfolio Return	0,0086	0,00887	0,008	0,04	0,01145
Optimal Portfolio Variance	0,00345	0,001745	0,001788	0,001775	0,001778
Portfolio Standard Deviation	0,056755	0,042388	0,041776	0,041297	0,042355

Figure 4 shows the Efficient Frontier, the Tangent Portfolio, the Global Minimum Variance portfolio, the two equally weighted portfolios and the Market Portfolio. So we reach the same conclusion regarding the index portfolio and the two equally weighted portfolios. We can obtain a greater return with the same level of risk on the efficient frontier portfolios.

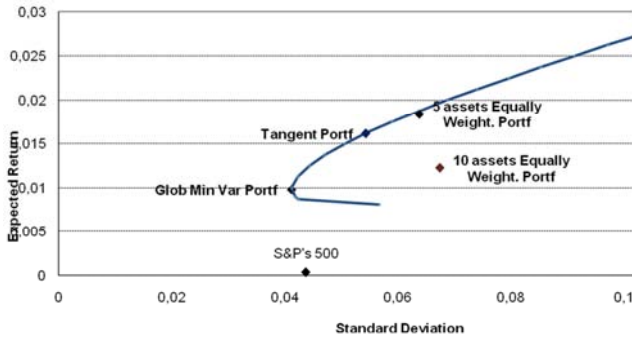


Figure 4. Efficient frontier and some relevant portfolios (short selling constraint).

Let's determine now the Efficient Frontier with the risk free asset and the Tangent Portfolio. In order to have an approximation for the risk free rate, we use the *US INTERBANK (1 MTH) interest rates* time series.

The risk free rate is obtained by:

$$RF = \frac{\sum (y_t / 1200)}{n} \tag{9}$$

where  $y$  is the *US INT.(1 Month)* time series and  $n$  is the number of observations.

The agent optimal choice in this case will be related to its risk aversion coefficient. The optimal portfolio will include a risk-free investment and a risky investment with weights in the risky assets proportional to the risky assets weights in the Tangency Portfolio. The agent problem is:

$$\begin{aligned}
& \max_{\omega_p} E[R_p] - \frac{R_A}{2} Var[R_p] \\
& s.t. \quad R_p = \sum_{i=1}^{N+1} \omega_i R_i \quad \sum_{i=1}^{N+1} \omega_i = 1
\end{aligned} \tag{10}$$

where  $i=1$  identifies the risk-free investment. Solving this problem, we can find the optimal weights.

$$\begin{aligned}
\omega &= \frac{1}{R_A} \Sigma^{-1} (\mu - r_f) \frac{\mathbf{1}' \Sigma^{-1} (\mu - r_f)}{\mathbf{1}' \Sigma^{-1} (\mu - r_f)} \\
\omega &= \frac{1}{R_A} \mathbf{1}' \Sigma^{-1} (\mu - r_f) \frac{\Sigma^{-1} (\mu - r_f)}{\mathbf{1}' \Sigma^{-1} (\mu - r_f)} = \frac{1}{R_A} (B - A r_f) \omega_M
\end{aligned} \tag{11}$$

The Efficient Frontier equation is

$$\sigma_p = \frac{\mu - RF}{\sqrt{A - B * RF - 1 \Sigma^{-1} \mu' * RF + C * RF^2}} \tag{12}$$

where  $A, B, C$  are defined as before. So in this case, the Efficient Frontier is a straight line, no more a curve.

The weights of the Tangency Portfolio are defined as:

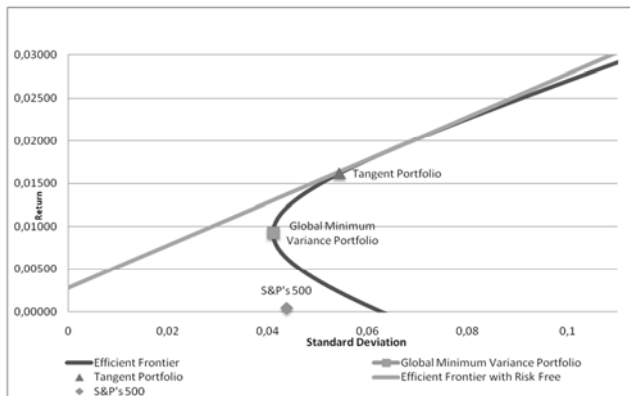
$$w = \frac{\Sigma^{-1} \mu' - RF * \Sigma^{-1} \mathbf{1}}{\mathbf{1}' \Sigma^{-1} \mu' - C * RF} \tag{13}$$

where ' $RF$ ' is the risk free rate computed before. So we can obtain the portfolio standard deviation and return with the usual formulas:

$$\begin{aligned}
\sigma_p &= (\omega' \Sigma \omega)^{1/2} \\
\mu_p &= \omega' \mu
\end{aligned} \tag{14}$$

Figure 5 shows the Efficient Frontier with the risk free asset (the straight line) and the Tangent portfolio. The

tangency portfolio in this case is the unique portfolio of the efficient frontier with the risk free asset that does not contain any investment in the risk free asset.



**Figure 5.** Efficient frontier with and without the risk free asset (short selling allowed).

## 4. Conclusion

The selected period for our analysis is characterized by a high volatility caused by adverse economic events for the U.S. market and certainly for the rest of global markets. We analyzed in this paper ten US stocks, part of the S&P's 500 index. Usually the basic idea is to take in account any stocks coming from different sectors in order to diversify as much as possible. This is due to the fact that usually our investor is risk averse and so a good differentiation is one of the first results we want achieve. In particular, we have selected these stocks: Hasbro, Coca Cola, Johnson & Johnson, Host Hotels and Resorts, Exxon Mobil, Edison INTL, Whirlpool, Titanium Metals, Bank of America and Red Hat. Stocks have been represented in the mean variance space and then we computed the correspondent Sharp Ratios, observing that the five stocks with the best performances are Hasbro, Coca Cola, Exxon, Titanium Metals and Red Hat. We observed a negative value for the BOA security, an evidence of a negative trend for the financial system. A negative Sharpe ratio indicates that a risk-less asset would perform better than the security being analyzed. Buying bank stocks during the crisis can be a good and profitable move.

Note that a portfolio is not simply a collection of individually good investments. Agents would like to invest in an efficient portfolio, that is, one in which there is no other portfolio that offers a greater return with the same or less risk, or that offers less risk with the same or greater expected return. A portfolio is said to be efficient if there is no portfolio having the same standard deviation with a greater expected return and there is no portfolio having the same return with a lower standard deviation. So, we built the Efficient Frontier, the collection of all efficient portfolios. The five assets collection we decided to keep is the one made in accordance with the Sharp Ratios criteria, because the relative Efficient Frontier is the one which is in the highest upper-left position in the mean-variance space.

In the case of allowed short selling, we observed 9 basis points of expected return and 41 basis points of standard deviation for the Global Minimum Variance portfolio and 16 basis points of expected return and 54 of standard deviation for the Tangent Portfolio. The Tangent Portfolio obviously has the best performance. The tangency portfolio combines the optimal combination of risky assets with a risk-free asset. The Global Minimum Variance portfolio is preferred by investors who are extremely risk averse. The Index Market portfolio lies on the dominated part of the Efficient Frontier, therefore for the same level of risk it is possible to obtain a higher return, so the Index Market portfolio is not efficient. We tested the efficiency of two equally weighted portfolios with 5 and 10 stocks. Both portfolios have a lower expected return than the efficient frontier portfolios with the same risk (standard deviation). We reach the same conclusions when we impose the short selling constraint. This is a more realistic case. In both cases, the Global Minimum Variance portfolio is composed mainly by stocks with the lowest variance (e.g.: Coca Cola with more than 50 percent).

Finally, we implement the Efficient Frontier with the risk free asset using as proxy the U.S. Treasury Bill, since it can be considered the real risk free asset (AAA rating). The straight line connecting the risk free return with the Tangent Portfolio is designated as the Capital Market Line. On one hand, investors who are extremely risk averse will select portfolio in the direction of the risk free asset, in order to have a lower expected return but also a lower variance. Investors who are only slightly risk averse will select a portfolio high up on the Capital Market Line possibly above the tangency (this is possible going short with the risk free asset). The basic idea is that to get higher expected return, one has to take more risk.

## References

- [1] Billio M., M. Getmansky, A. Lo and L. Pelizzon (2010), Econometric Measures of Systemic Risk in Finance and Insurance sectors, MIT WP 4774-10, NBER WP 16223, Journal of Financial Economics.
- [2] Bodie, Zvi, Alex Kane, and Alan J. Marcus, 2005, Investments, 6th edition, McGraw-Hill/Irwin.
- [3] Campbell, John Y., A. W. Lo, and A. C. MacKinlay. The Econometrics of Financial Markets. Princeton University Press, Princeton, NJ, 1997.
- [4] Chavas, Jean-Paul, and M. T. Holt. "Economic Behavior under Uncertainty: A Joint Analysis of Risk Preferences and Technology" Review of Economics and Statistics.
- [5] DeGroot, Morris H. Optimal Statistical Decisions. McGraw-Hill, New York, 1970.
- [6] Dixit, Avinash K., and Robert S. Pindyck. Investment under Uncertainty. Princeton University Press, 1994.
- [7] Dre'ze, Jacques. Essays on Economic Decisions under Uncertainty. Cambridge University Press, New York, 1987.

- [8] Epstein, L., and S. Zin. "Substitution, Risk Aversion, and Temporal Behavior of Consumption and Asset Returns: An Empirical Investigation" *Journal of Political Economy*. 99(1991): 263–286.
- [9] Elton, E. J., Martin, J. G., Brown, S. J., Goetzmann, W. N. (2014), *Modern Portfolio Theory and Investment Analysis*. United States: Wiley.
- [10] Hirshleifer, Jack, and John G. Riley. *The Analytics of Uncertainty and Information*. Cambridge University Press, Cambridge, 1992.
- [11] Hull, John. *Options, Futures and Other Derivatives*. Fifth Edition. Prentice Hall, 2002.
- [12] J. F. Affleck-Graves and A. H. Money 1976, A comparison of two portfolio selection models, *The Investment Analysts Journal*, 7(4), 1976, 35-40.
- [13] Kahneman, Daniel, and A. Tversky. "Prospect Theory: An Analysis of Decision under Risk" *Econometrica* 47(1979): 263–191.
- [14] Low, R. K. Y.; Faff, R.; Aas, K. (2016). "Enhancing mean–variance portfolio selection by modeling distributional asymmetries". *Journal of Economics and Business*.
- [15] Mandelbrot, B. (2004). *The (Mis) Behavior of Markets*. New York: Basic Books.
- [16] Markowitz, H. "Portfolio Selection" *Journal of Finance*. 6(1952): 77–91.
- [17] Menezes, C., and D. Hanson. "On the Theory of Risk Aversion" *International Economic Review*. 11(1970): 481–487.
- [18] Parigi B. and L. Pelizzon (2008), *Diversification and Ownership Structure*, CESifo Working Paper 1590. *Journal of Banking and Finance*, 32, 9, 1709-1722.
- [19] Pelizzon L and G. Weber (2005) *Efficient Portfolios Conditional on Housing: Evidence from the UCI survey*, Trends in Saving and Wealth n.9/2005, Pioneer Investments.
- [20] Pelizzon L and G. Weber (2008) *Optimal portfolio composition when real assets and liabilities are taken into account*, Trends in Saving and Wealth n.1/2008, Pioneer Investments.
- [21] Pratt, John W. "Risk Aversion in the Small and in the Large" *Econometrica* 32(1964): 122–136.
- [22] Savage, Leonard J. *The Foundations of Statistics*. Wiley, New York, 1954.
- [23] Sharpe, William F. 1966. "Mutual Fund Performance." *Journal of Business*. January, 39, pp. 119–38.
- [24] Tobin, James. 1958. "Liquidity Preference as Behavior Towards Risk." *Review of Economic Studies*, February.